

Spectral Properties of a Binomial Matrix

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1 Introduction

In [3], Peele and Stănică studied $n \times n$ matrices with the (i, j) entry the binomial coefficient $\binom{i-1}{j-1}$ (matrix L_n), respectively $\binom{i-1}{n-j}$ (matrix R_n) and derived many interesting results on powers of these matrices. In [5], the author found that the same is true for a much larger class of what he called *netted matrices*, namely matrices with entries satisfying a certain type of recurrence among the entries of all 2×2 cells. In this paper we continue the work in [3, 5]. We find the generating function for all entries of R_n^e on each row or column and we find all eigenvalues (with their multiplicities) of R_n modulo 3 and 5. We also give a precise relation between R_n and L_n with the help of a permutation matrix and we prove the conjecture stated in [3].

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2 Some Generating Functions

We denote by $a_{i,j} = \binom{i-1}{n-j}$. We observe that they satisfy the recurrence

$$a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}, \quad (1)$$

which can be extended for $i \geq 0, j \geq 0$, using the boundary conditions $a_{1,n} = 1$, $a_{1,j} = 0$, $j \neq n$. We shall use the following consequences of the boundary conditions and recurrence (1): $a_{i,j} = 0$ for $i + j \leq n$, and $a_{i,n+1} = 0$, $1 \leq i \leq n$. In [3] it was proved that the entries of R_n^e satisfy the recurrence

$$F_{e-1}a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1}a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)}, \quad (2)$$

where F_e is the Fibonacci sequence. We were unable to find closed forms for *all* entries of R_n^e , however we found generating functions for the entries in each row and column of R_n^e . We use the following lemma, which can be proven easily once one guesses the two formulas (see also [3, 5]).

Lemma 1. *The elements on the first row and the first column of R_n^e are given by*

$$\begin{aligned} a_{1,j}^{(e)} &= \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} \\ a_{i,1}^{(e)} &= F_{e-1}^{n-i} F_e^{i-1}. \end{aligned} \quad (3)$$

We extend the tableau $a_{i,j}^{(e)}$, for $i \geq 0, j \geq 0$, using the recurrence (2) and Lemma 1.

Theorem 2. *The generating function for the i -th row of R_n^e is*

$$r_i^{(e)}(x) = \sum_{j \geq 1} a_{i,j}^{(e)} x^{j-1} = (F_e + F_{e+1}x)^{i-1} (F_{e-1} + F_e x)^{n-i}$$

and, if $e > 1$, the generating function for the j -th column of R_n^e is

$$c_j^{(e)}(x) = \left(\frac{F_{e+1}x - F_e}{F_{e-1} - F_e x} \right)^{j-1} \frac{F_{e-1}^n}{F_{e-1} - F_e x} \left[1 + \sum_{s=1}^{j-1} \binom{n}{s} \left(\frac{F_e(F_{e-1} - F_e x)}{F_{e-1}(F_{e+1}x - F_e)} \right)^s \right].$$

and $c_j^{(1)}(x) = (1-x)^{j-1-n}x^{n-j}$.

Proof. Multiplying (2) by x^{j-1} and summing for $j \geq 1$, we get

$$\begin{aligned}
& F_{e-1} \sum_{j \geq 1} a_{i,j}^{(e)} x^{j-1} + F_e \sum_{j \geq 1} a_{i,j-1}^{(e)} x^{j-1} \\
& \quad = F_e \sum_{j \geq 1} a_{i-1,j}^{(e)} x^{j-1} + F_{e+1} \sum_{j \geq 1} a_{i-1,j-1}^{(e)} x^{j-1} \iff \\
& F_{e-1} r_i^{(e)}(x) + F_e x r_i^{(e)}(x) + F_e a_{i,0}^{(e)} = F_e r_{i-1}^{(e)}(x) + F_{e+1} x r_{i-1}^{(e)}(x) + F_{e+1} a_{i-1,0}^{(e)} \iff \\
& (F_{e-1} + F_e x) r_i^{(e)}(x) - (F_e + F_{e+1} x) r_{i-1}^{(e)}(x) = F_{e+1} a_{i-1,0}^{(e)} - F_e a_{i,0}^{(e)} \\
& = F_{e-1} a_{i,1}^{(e)} - F_e a_{i-1,1}^{(e)} \stackrel{\text{Lemma 1}}{=} F_{e-1} F_{e-1}^{n-i} F_e^{i-1} - F_e F_{e-1}^{n-i+1} F_e^{i-2} = 0.
\end{aligned}$$

Therefore,

$$(F_{e-1} + F_e x) r_i^{(e)}(x) = (F_e + F_{e+1} x) r_{i-1}^{(e)}(x),$$

which implies

$$r_i^{(e)}(x) = \left(\frac{F_e + F_{e+1} x}{F_{e-1} + F_e x} \right)^{i-1} r_1^{(e)}(x).$$

The generating function for the first row of R_n^e is

$$\begin{aligned}
r_1^{(e)}(x) &= \sum_{j \geq 1} \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} x^{j-1} \\
&= \sum_{s \geq 0} \binom{n-1}{s} F_{e-1}^{n-s-1} F_e^s x^s \\
&= F_{e-1}^{n-1} \sum_{s \geq 0} \binom{n-1}{s} \left(\frac{F_e x}{F_{e-1}} \right)^s \\
&= F_{e-1}^{n-1} \left(1 + \frac{F_e x}{F_{e-1}} \right)^{n-1} = (F_{e-1} + F_e x)^{n-1}.
\end{aligned}$$

Thus,

$$r_i^{(e)}(x) = \left(\frac{F_e + F_{e+1} x}{F_{e-1} + F_e x} \right)^{i-1} (F_{e-1} + F_e x)^{n-1} = (F_e + F_{e+1} x)^{i-1} (F_{e-1} + F_e x)^{n-i}.$$

and the first claim is proved.

It is trivial to find the generating function of the columns of R_n . Now, for $e > 1$, multiplying (1) by x^{i-1} and summing for $i \geq 1$, we get

$$\begin{aligned}
& F_{e-1} \sum_{i \geq 1} a_{i,j}^{(e)} x^{i-1} + F_e \sum_{i \geq 1} a_{i,j-1}^{(e)} x^{i-1} \\
& \quad = F_e \sum_{i \geq 1} a_{i-1,j}^{(e)} x^{i-1} + F_{e+1} \sum_{i \geq 1} a_{i-1,j-1}^{(e)} x^{i-1} \iff \\
& F_{e-1} c_j^{(e)}(x) + F_e c_{j-1}^{(e)}(x) = F_e x c_j^{(e)}(x) \\
& \quad + F_{e+1} x c_{j-1}^{(e)}(x) + F_e a_{0,j}^{(e)} + F_{e+1} a_{0,j-1}^{(e)} \iff \\
& (F_{e-1} - F_e x) c_j^{(e)}(x) + (F_e - F_{e+1} x) c_{j-1}^{(e)}(x) \\
& = F_e a_{0,j}^{(e)} + F_{e+1} a_{0,j-1}^{(e)} = F_{e-1} a_{1,j}^{(e)} + F_e a_{1,j-1}^{(e)} \\
& \stackrel{\text{Lemma 1}}{=} F_{e-1} \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1} + F_e \binom{n-1}{j-2} F_{e-1}^{n-j+1} F_e^{j-2} \\
& = \binom{n}{j-1} F_{e-1}^{n-j+1} F_e^{j-1}.
\end{aligned}$$

If $e > 1$, the generating function for the first column of R_n^e is

$$\begin{aligned}
c_1^{(e)}(x) &= \sum_{i \geq 1} F_{e-1}^{n-i} F_e^{i-1} x^{i-1} = F_{e-1}^{n-1} \sum_{s \geq 0} \left(\frac{F_e x}{F_{e-1}} \right)^s \\
&= F_{e-1}^{n-1} \frac{1}{1 - \frac{F_e x}{F_{e-1}}} = \frac{F_{e-1}^n}{F_{e-1} - F_e x}.
\end{aligned}$$

It is not difficult to obtain that a recurrence of the form

$$\alpha c_j + \beta c_{j-1} = u_{j-1},$$

has the solution

$$\begin{aligned}
c_j &= \left(\frac{-\beta}{\alpha} \right)^{j-2} c_1 + \frac{1}{\alpha} \left(u_{j-1} - \frac{\beta}{\alpha} u_{j-2} + \cdots + \left(\frac{-\beta}{\alpha} \right)^{j-2} u_1 \right) \\
&= \left(\frac{-\beta}{\alpha} \right)^{j-2} c_1 + \frac{1}{\alpha} \sum_{s=1}^{j-1} u_s \left(\frac{-\beta}{\alpha} \right)^{j-s-1}.
\end{aligned} \tag{4}$$

Using (4) in the recurrence for $c_j^{(e)}(x)$, we get

$$\begin{aligned} c_j^{(e)}(x) &= \left(\frac{F_{e+1}x - F_e}{F_{e-1} - F_ex} \right)^{j-1} c_1^{(e)}(x) \\ &\quad + \frac{1}{F_{e-1} - F_ex} \sum_{s=1}^{j-1} \binom{n}{s} \left(\frac{F_{e+1}x - F_e}{F_{e-1} - F_ex} \right)^{j-s-1} F_{e-1}^{n-s} F_e^s \\ &= \left(\frac{F_{e+1}x - F_e}{F_{e-1} - F_ex} \right)^{j-1} \frac{F_{e-1}^n}{F_{e-1} - F_ex} \left(1 + \sum_{s=1}^{j-1} \binom{n}{s} \left(\frac{F_e(F_{e-1} - F_ex)}{F_{e-1}(F_{e+1}x - F_e)} \right)^s \right). \end{aligned}$$

□

3 Characteristic Polynomials Modulo 3

In this section, we are interested in the eigenvalues of R_n modulo p , and their multiplicities.

Lemma 3. *We have*

$$\text{trace}(R_n) = F_n.$$

Proof. We need to prove

$$\sum_{i=1}^n \binom{i-1}{n-i} \stackrel{n-i=k}{=} \sum_{k=0}^n \binom{n-k-1}{k} = F_n,$$

which is a well-known relation [1].

□

Our main result of this section is

Theorem 4. *The characteristic polynomial of R_n modulo 3, say $p_n(x)$, is*

$$\begin{aligned} p_{4k}(x) &= (2 + x + x^2)^k (2 + 2x + x^2)^k = (1 + x^4)^k \\ p_{4k+1}(x) &= 2(1 + x)^{2[\frac{k+1}{2}]} (2 + x)^{2[\frac{k}{2}]+1} (1 + x^2)^k \\ &= -(x^4 - 1)^k (x - 1) \text{ if } k \text{ even or } -(x^4 - 1)^k (x + 1) \text{ if } k \text{ odd} \\ p_{4k+2}(x) &= (2 + x + x^2)^{2[\frac{k+1}{2}]} (2 + 2x + x^2)^{2[\frac{k}{2}]+1} \end{aligned}$$

$$\begin{aligned}
&= (x^4 + 1)^k(x^2 - x - 1) \text{ if } k \text{ even or } (x^4 + 1)^k(x^2 + x - 1) \text{ if } k \text{ odd} \\
p_{4k+3}(x) &= 2(1+x)^{2[\frac{k}{2}]+1}(2+x)^{2[\frac{k+1}{2}]}(1+x^2)^{k+1} \\
&= -(x^4 - 1)^k(x+1)(x^2+1) \text{ if } k \text{ even or} \\
&\quad -(x^4 - 1)^k(x-1)(x^2+1) \text{ if } k \text{ odd.}
\end{aligned}$$

Proof. First, we prove that $R_{2k}^4 \equiv -I_{2k} \pmod{3}$ and $R_{2k+1}^4 \equiv I_{2k+1} \pmod{3}$. In fact a more general result is true; using Theorem 12 of [3] we know that if $p|F_{p+1}$, then $R_{2k}^{p+1} \equiv -I_{2k} \pmod{p}$ and $R_{2k+1}^{p+1} \equiv I_{2k+1} \pmod{p}$. In our case, $3|F_4 = 3$. Thus, the minimal polynomial of R_n modulo 3 equals $x^4 + 1$ for n even and $x^4 - 1$ for n odd. Since $\gcd(3, 4) = 1$, we deduce that $x^4 \pm 1$ has distinct roots. Thus, R_n is diagonalizable (over a degree 2 extension of \mathbf{Z}_3).

First, $n = 4k$. Thus, the minimal polynomial of R_n modulo 3 equals $x^4 + 1 = (x^2 - x - 1)(x^2 + x - 1)$, which has the solutions $\alpha, \beta, -\alpha, -\beta$ (where $\beta = 1 - \alpha$) in the quadratic extension of \mathbf{Z}_3 , namely $\mathbf{Z}_3[x]/(x^2 + x - 1)$. Let a, b, c, d be the multiplicities of $\alpha, \beta, -\alpha, -\beta$, respectively. We want to show that $a = b = c = d = k$. Since R_n has dimension $n = 4k$,

$$a + b + c + d = 4k.$$

Now, using Lemma 3, we get

$$a\alpha + b\beta - c\alpha - d\beta \equiv F_{4k} \pmod{3}.$$

It is known (see [1]) that $F_n|F_{nk}$. Thus, F_{4k} is divisible by $F_4 = 3$, so $F_{4k} \equiv 0 \pmod{3}$.

Therefore, $a = c$ and $b = d$, so $a + b = c + d = 2k$. The eigenvalues of R_{4k} correspond to eigenvalues of R_{4k}^3 and since, $R_{4k}^3 = -R_{4k}^{-1} \pmod{3}$, we get that α corresponds to α^3 ,

which corresponds to $-\frac{1}{\alpha} = \beta$, and similarly, $\beta \implies \beta^3 \implies -\frac{1}{\beta} = \alpha$. We deduce $a = b$ and so, $a = b = c = d = k$.

In a similar manner we can show the other cases. \square

With the same method, we can prove

Theorem 5. *The characteristic polynomial of R_n modulo 5, say $p_n(x)$, is*

$$\begin{aligned} p_{4k}(x) &= (x-2)^{4k}; & p_{4k+1}(x) &= -(x-1)^{4k+1} \\ p_{4k+2}(x) &= (x+2)^{4k+2}; & p_{4k+3}(x) &= -(x+1)^{4k+3} \end{aligned}$$

The interesting fact is that we were able to find, after quite a bit of work, the eigenvalues (with their multiplicities), for all the first few primes p we considered.

4 Characteristic Polynomial, Eigenvalues and Eigenvectors of R_n

In [3], the authors proposed a conjecture on the eigenvalues. No pattern leaped out from the eigenvectors' set, if one looks at the first few values. In this section we prove the conjecture and we find the eigenvectors of R_n . In [3], we showed

Lemma 6. *The inverse of R_n is the matrix*

$$R_n^{-1} = \left((-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1 \leq i, j \leq n}.$$

We define K_n to be the matrix with (i, j) -entry $\delta_{i, n-j+1}$ (the Kronecker symbol).

Remark 7. *We remark that K_n is the permutation matrix having 1 on the secondary diagonal and 0 elsewhere.*

Theorem 8. Denote $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. The eigenvalues of R_n are:

1. $\{(-1)^{k+i}\phi^{2i-1}, (-1)^{k+i}\bar{\phi}^{2i-1}\}_{i=1,\dots,k}$, if $n = 2k$.
2. $\{(-1)^k\} \cup \{(-1)^{k+i}\phi^{2i}, (-1)^{k+i}\bar{\phi}^{2i}\}_{i=1,\dots,k}$, if $n = 2k + 1$.

Proof. First, we show that R_n is a permutation matrix away from L_n , namely

$$R_n \cdot K_n = L_n \iff R_n = L_n \cdot K_n. \quad (5)$$

It is a trivial matter to prove $K_n^2 = I_n$, which will give the equivalence. It suffices to show the second identity, which follows easily, since an entry in $L_n \cdot K_n$ is

$$\sum_{k=1}^n \binom{i-1}{k-1} \delta_{k,n-j+1} = \binom{i-1}{n-j}.$$

Now, denote by A_n , the matrix obtained by taking absolute values of entries of R_n^{-1} .

We show that R_n has the same characteristic polynomial (eigenvalues) as A_n , namely we prove their similarity,

$$K_n \cdot R_n \cdot K_n = A_n. \quad (6)$$

Since $K_n \cdot R_n \cdot K_n = K_n \cdot L_n$ (by (5)), to show (6) it suffices to prove that $K_n \cdot L_n =$

$\left(\binom{n-i}{j-1}\right)_{i,j}$. Therefore, we need

$$\sum_{k=1}^n \delta_{i,n-k+1} \binom{k-1}{j-1} = \binom{n-i}{j-1},$$

which is certainly true.

We use a result of [2] to show that A_n in turn is similar to D_n , the diagonal matrix whose diagonal entries are the elements in the eigenvalues set listed in decreasing order

according to size of the absolute value. For instance, for $n = 4$,

$$D_4 = \begin{pmatrix} \alpha^3 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta^3 \end{pmatrix}$$

If A_n is similar to D_n , the theorem will be proved. We sketch here the argument of [2]

for the convenience of the reader. Define an array $b_{n,m}$ by

$$b_{n,0} = 1 \text{ for all } n \geq 0$$

$$b_{n,m} = 0 \text{ for all } m > n$$

$$b_{n,m} = b_{n-1,m-1} (-1)^m \frac{F_n}{F_m} \text{ for all } m \leq n$$

Except for the sign, $|b_{n,m}| = \frac{F_n F_{n-1} \cdots F_{n-m+1}}{F_m \cdots F_1}$. Let $C_n = (c_{i,j})_{i,j}$, where

$$\begin{cases} c_{i,i+1} = 1 & \text{if } i = 1, \dots, n-1 \\ c_{n,j} = -b_{n,n+1-j} & \text{if } j = 1, \dots, n \\ c_{i,j} = 0 & \text{otherwise.} \end{cases}$$

We observe that, in fact, C_n is the companion matrix of the polynomial with coefficients $b_{n,n+1-j}$. Let X_n be the matrix with entries $\binom{n-i}{j-1} F_{i-2}^{j-1} F_{i-1}^{n-j}$. It turns out that the eigenvector matrix E_n of A_n , with columns vectors listed in decreasing order of absolute value of the corresponding eigenvalues, normalized so that the last row is made up of all 1's, satisfies

$$X_n E_n = V_n,$$

where V_n is the Vandermonde matrix, which is the eigenvector matrix of C_n with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues.

Also,

$$X_n A_n X_n^{-1} = C_n \text{ and } E_n^{-1} A_n E_n = D_n.$$

Our theorem follows. □

Easily we deduce

Corollary 9. *The eigenvectors matrix of R_n , say W_n , with eigenvectors listed in decreasing order of the absolute values of the corresponding eigenvalues, is*

$$W_n = K_n E_n = K_n X_n^{-1} V_n.$$

Proof. We showed that

$$K_n R_n K_n = A_n \text{ and } E_n^{-1} A_n E_n = D_n.$$

It follows that $(E_n^{-1} K_n) R_n (K_n E_n) = D_n$, which together with $X_n E_n = V_n$, proves the corollary. \square

Example 10. *For $n = 4$, the eigenvectors matrix is*

$$W_4 = \begin{pmatrix} -\alpha^3 & \alpha & \beta & -\beta^3 \\ \alpha^2 & -\frac{1}{3}\beta & -\frac{1}{3}\alpha & \beta^2 \\ -\alpha & -\frac{1}{3}\alpha^2 & -\frac{1}{3}\beta^2 & -\beta \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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